Math 1B – Calculus – Test 3 – Spring '10   Name____________________________

Write all responses on separate paper. Show your work for credit.

1. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

   a. $\int_{1/2}^{1} \frac{1}{\sqrt{4x^2-1}}\,dx$ b. $\int_{1}^{\infty} \frac{\ln x}{x^2}\,dx$

2. Use the comparison theorem to determine whether the integral is convergent or divergent: $\int_{1}^{2} \frac{e^x}{x\sqrt{x-1}}\,dx$

3. Evaluate the integral $\int_0^{\infty} \sin(nx)e^{-x}dx$ in terms of n.

4. Find the length of the curve $y = \frac{1}{9}\left(x^2+6\right)^{3/2}$, $0 \le x \le 1$.

5. Find the length of the curve $y = \int_0^x \sqrt{\cos^2 t - 1}\,dt$, $0 \le x \le \pi$.

6. Find the surface area generated by rotating about the y–axis the curve $x = \sin(y)$ for $0 \le y \le \pi$.

7. Find the surface area of surface generated by rotating the curve $y = x^{-2}$ about the x-axis for $x \ge 1$.

8. A circular gate in an irrigation canal has diameter 5 meters at the top. If the canal is filled to a depth of 2 meters below the top of the gate, find the hydrostatic force on one side of the gate.

9. Find the centroid of the region bounded by $y = \sin(\pi x)$ and $y = x(x-1)$, $0 \le x \le ½$.

10. Find the moments and center of mass of the system of objects that have masses 4, 7 and 11 at the points (–2,3), (1,1) and (3,–2), respectively.

11. Lengths of human pregnancies are normally distributed with mean 268 days and standard deviation 15 days. What percentage of pregnancies last between 250 days and 280 days? Hint: use the normal probability density function: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$

12. Let $f(x) = kx(1-x^2)$ if $0 \le x \le 1$ and $f(x) = 0$ if $x < 0$ or $x > 1$.

    a. For what value of k is $f(x)$ a probability density function?
    b. For that value of k, find $P(X \ge 0)$
    c. Find the mean.

Math 1B – Calculus – Test 3 Solutions – Spring '10

1. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

a. $\int_{1/2}^{1} \frac{1}{\sqrt{4x^2-1}}\,dx$ b. $\int_{1}^{\infty} \frac{\ln x}{x^2}\,dx$

SOLN:

$$\int_{1/2}^{1} \frac{1}{\sqrt{4x^2-1}}\,dx = \frac{1}{2}\int_{0}^{\pi/3} \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta-1}}\,d\theta = \frac{1}{2}\int_{1/2}^{1} \sec\theta d\theta$$

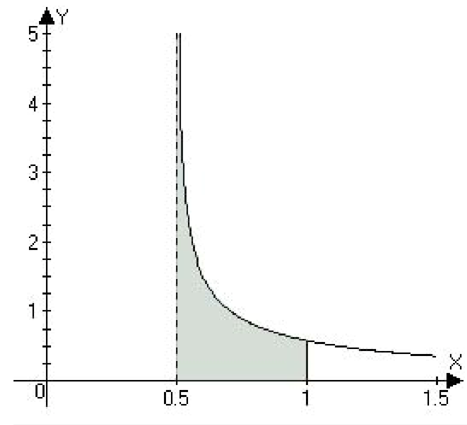
$$= \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right|\Big|_{0}^{\pi/3} = \frac{1}{2}\ln\left|2+\sqrt{3}\right|$$

So it's covergent and we know it's value. Compare

$$0 < \frac{1}{\sqrt{4x^2-1}} = \frac{1}{\sqrt{2x+1}\sqrt{2x-1}} \le \frac{1}{\sqrt{2}\sqrt{2x-1}} < \frac{1}{\sqrt{2x-1}}$$

on [½,1] to see the integral is convergent since

$$\int_{1/2}^{1} \frac{dx}{\sqrt{2x-1}} = \sqrt{2x-1}\Big|_{1/2}^{1} = 1$$



SOLN: $\int_{1}^{\infty} \frac{\ln x}{x^2}\,dx = \int_{0}^{\infty} ue^{-u}\,du = \lim_{b\to\infty} -ue^{-u}\Big|_{0}^{b} + \int_{0}^{\infty} e^{-u}du = 0 - \lim_{b\to\infty} -e^{-u}\Big|_{0}^{b} = 1$ You can also do i.by.p here.

2. Use the comparison theorem to determine whether the integral is convergent or divergent:

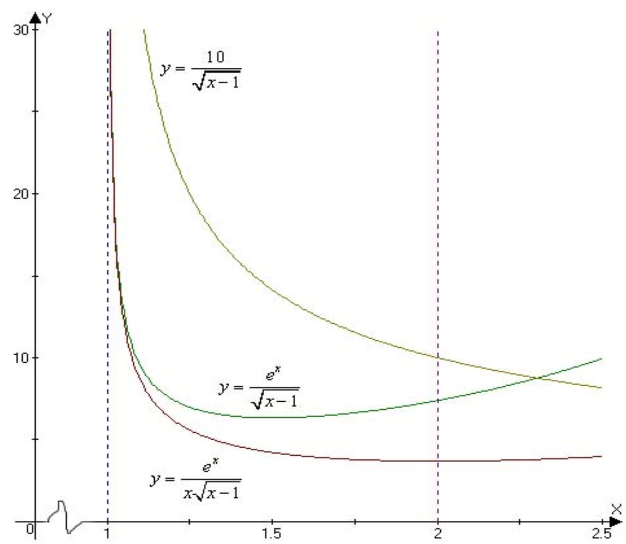
$$\int_{1}^{2} \frac{e^x}{x\sqrt{x-1}}\,dx$$

SOLN:

$$0 < \int_{1}^{2} \frac{e^x}{x\sqrt{x-1}}\,dx < \int_{1}^{2} \frac{e^2}{\sqrt{x-1}}\,dx < \int_{1}^{2} \frac{10}{\sqrt{x-1}}\,dx,$$

$$= \int_{0}^{1} \frac{10}{\sqrt{u}}\,du = \lim_{b\to 0^+} 20\sqrt{u}\Big|_{b}^{1} = 20$$

therefore the integral is convergent.



3. Evaluate the integral $\int_{0}^{\infty} \sin(nx)e^{-x}dx$ in terms of *n*.

SOLN:

$$\int_{0}^{\infty} \sin(nx)e^{-x}dx = -e^{-x}\sin(nx)\Big|_{0}^{\infty} + n\int_{0}^{\infty}\cos(nx)e^{-x}dx = 0 - ne^{-x}\cos(nx)\Big|_{0}^{\infty} - n^2\int_{0}^{\infty}\sin(nx)e^{-x}dx$$

$$\left(1+n^2\right)\int_{0}^{\infty}\sin(nx)e^{-x}dx = n \Leftrightarrow \int_{0}^{\infty}\sin(nx)e^{-x}dx = \frac{n}{1+n^2}$$

4. Find the length of the curve $y = \frac{1}{9}\left(x^2+6\right)^{3/2}$, $0 \le x \le 1$.

SOLN: $y' = \frac{x}{3}\left(x^2+6\right)^{1/2} \Rightarrow 1+(y')^2 = 1+\frac{x^2}{9}\left(x^2+6\right) = \frac{x^4+6x^2+9}{9} = \left(\frac{x^2+3}{3}\right)^2$

$$L = \int_{0}^{1}\sqrt{1+(y')^2}\,dx = \frac{1}{3}\int_{0}^{1}\left(x^2+3\right)dx = \frac{1}{3}\left(\frac{x^3}{3}+3x\right)\Bigg|_{0}^{1} = \frac{10}{9}$$

5. Find the length of the curve $y=\int_0^x \sqrt{\cos^2 t-1}\,dt$, $0\le x\le \pi$.

SOLN: $y'=\sqrt{\cos^2 x-1}\Rightarrow 1+(y')^2=\cos^2 x\Rightarrow L=\int_0^{\pi}\sqrt{\cos^2 x}\,dx=\int_0^{\pi}|\cos x|\,dx=2\int_0^{\pi/2}\cos x\,dx=2$

6. Find the surface area generated by rotating about the y–axis the curve $x = \sin(y)$ for $0 \le y \le \pi$. First substitute $u = \cos(y)$ so that $du = -\cos(y)dy$ and $1>u>-1$ then reverse the bounds and substitute $u = \tan\theta$ so that $du = \sec^2(\theta)d\theta$. Now use symmetry about the y-axis and double the integral from $0 \le \theta \le \pi/4$.

SOLN:
$$2\pi\int_0^{\pi}\sin y\sqrt{1+\cos^2 y}\,dy=-2\pi\int_1^{-1}\sqrt{1+u^2}\,du=2\pi\int_{-\pi/4}^{\pi/4}\sec^3\theta\,d\theta$$
$$=2\pi\left(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4}=2\pi\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right)$$

7. Find the surface area of surface generated by rotating the curve $y = x^{-2}$ about the x-axis for $x \ge 1$.

SOLN: $2\pi\int_1^{\infty}r\,ds=2\pi\int_1^{\infty}\frac{1}{x^2}\sqrt{1+\frac{4}{x^6}}\,dx=2\pi\int_1^{\infty}\frac{\sqrt{x^6+4}}{x^5}\,dx$. This is worth a trig substitution.

$x^3 = 2\tan\theta$ yields $2\pi\int_1^{\infty}\frac{\sqrt{x^6+4}}{x^5}\,dx=2\pi\int_1^{\infty}\frac{4\sec^3\theta}{3x^3}\,d\theta$ but there doesn't seem to be a nice way to substitute for the x^3…so there isn't a nice antiderivative, outside of the Elliptic functions. So try integrating over y: $2\pi\int_1^{1}r\,ds=2\pi\int_0^{1}y\sqrt{1+\frac{1}{4y^3}}\,dy=\pi\int_0^{1}\sqrt{\frac{4y^3+1}{y}}\,dy$. Unfortunately, this is also difficult because of the lack of an elementary antiderivative. In a situation like this, we are forced to employ numerical approximations.

$$\int\frac{\sqrt{x^6+4}}{x^5}\,dx=$$

$$\frac{1}{8x^4\sqrt{x^6+4}}\left(-2\left(x^6+4\right)+\sqrt[6]{-2}\;3^{3/4}\,x^4\sqrt{-\sqrt[6]{-1}\left(\sqrt[3]{2}\,x^2+2(-1)^{2/3}\right)}\right.$$

$$\sqrt{(-2)^{2/3}x^4+2\sqrt[3]{-2}\,x^2+4}$$

$$\left.F\left(\sin^{-1}\left(\frac{\sqrt{\left(-i+\sqrt{3}\right)\left(\sqrt[3]{2}\,x^2+2\right)}}{2\sqrt[4]{3}}\right)\middle|\sqrt[3]{-1}\right)\right)+\text{constant}$$

8. A circular gate in an irrigation canal has diameter 5 meters. If the canal is filled to a depth of 2 meters below the top of the gate, find the hydrostatic force on one side of the gate.
SOLN: Center the coordinate system at the center of the gate. Then
weight density * depth * element of area = 9800 * (3 – y) * sqrt(25 – $4x^2$)

Thus the total fluid force against the gate is

$$9800\int_{-2.5}^{0.5}(3-y)\sqrt{25-4y^2}\,dy = 9800\left[(3)\int_{-2.5}^{0.5}\sqrt{25-4y^2}\,dy - \int_{-2.5}^{0.5}y\sqrt{25-4y^2}\,dy\right].$$

Now,

$$\int_{-2.5}^{0.5}\sqrt{25-4y^2}\,dy = \frac{25}{2}\int_{-\pi/2}^{\arcsin 0.2}\cos^2\theta\,d\theta = \frac{25}{2}\int_{-\pi/2}^{\arcsin 0.2}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{25}{2}\left(\frac{\theta}{2}+\frac{\sin\theta\cos\theta}{2}\right)\Bigg|_{-\pi/2}^{\arcsin 0.2} = \frac{25}{4}\left(\frac{\pi}{2}+\arcsin\left(\frac{1}{5}\right)+\frac{2\sqrt{6}}{25}\right)$$

and $\int_{-2.5}^{0.5}y\sqrt{25-4y^2}\,dy = -\frac{1}{8}\int_{0}^{24}\sqrt{u}\,du = -\frac{1}{12}(24)^{3/2} = -4\sqrt{6}$. So, all together we've got

$$9800\left[\frac{75}{4}\left(\frac{\pi}{2}+\arcsin\left(\frac{1}{5}\right)+\frac{2\sqrt{6}}{25}\right)+4\sqrt{6}\right] = 2450\left[\frac{75\pi}{2}+75\arcsin\left(\frac{1}{5}\right)+22\sqrt{6}\right]$$

9. Find the centroid of the region bounded by $y = \sin(\pi x)$ and $y = x(x-1)$, $0 \le x \le ½$.

SOLN: The mass is $M = \int_0^{1/2}\sin(\pi x) - x(x-1)\,dx = \frac{-\cos(\pi x)}{\pi} - \frac{x^3}{3} + \frac{x^2}{2}\Bigg|_0^{1/2} = \frac{1}{\pi} - \frac{1}{24} + \frac{1}{8} = \frac{1}{\pi} + \frac{1}{12}$

$$M_y = \int_0^{1/2} x\sin(\pi x) - x^2(x-1)\,dx = \int_0^{1/2} x\sin(\pi x)\,dx - \int_0^{1/2} x^2(x-1)\,dx$$

$$= \frac{-x}{\pi}\cos(\pi x)\Bigg|_0^{1/2} + \frac{1}{\pi}\int_0^{1/2}\cos(\pi x)\,dx - \left(\frac{x^4}{4}-\frac{x^3}{3}\right)\Bigg|_0^{1/2} = 0 + \frac{\sin(\pi x)}{\pi^2}\Bigg|_0^{1/2} - \frac{1}{64} + \frac{1}{24}$$

$$= \frac{1}{\pi^2} + \frac{5}{192}$$

$$M_x = \frac{1}{2}\int_0^{1/2}\sin^2(\pi x) - x^2(x-1)^2\,dx = \frac{1}{2}\int_0^{1/2}\frac{1-\cos(2\pi x)}{2}\,dx - \int_0^{1/2} x^4 - 2x^3 + x^2\,dx$$

$$= \frac{1}{2}\left(\frac{x}{2} - \frac{\sin(2\pi x)}{4\pi}\right)\Bigg|_0^{1/2} - \left(\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right)\Bigg|_0^{1/2} = \frac{1}{8} - \frac{1}{160} + \frac{1}{32} - \frac{1}{24} = \frac{13}{120}$$

So $\bar{x} = \frac{M_y}{M} = \frac{\frac{1}{\pi^2}+\frac{5}{192}}{\frac{1}{\pi}+\frac{1}{12}} = \frac{192+5\pi^2}{192\pi+16\pi^2} \approx 0.317$ and $\bar{y} = \frac{M_x}{M} = \frac{\frac{13}{120}}{\frac{1}{\pi}+\frac{1}{12}} = \frac{13\pi}{120+10\pi} \approx 0.270$

10. Find the moments and center of mass of the system of objects that have masses 4, 7 and 11 at the points (–2,3), (1,1) and (3,–2), respectively.

SOLN: $M = 4+7+11 = 22$. $M_y = \sum_{i=1}^{3} x_i m_i = -2(4)+1(7)+3(11) = 32$.

$M_x = \sum_{i=1}^{3} y_i m_i = 3(4)+1(7)-2(11) = -3$ so $\bar{x} = \frac{32}{22} = \frac{16}{11}$ and $\bar{y} = \frac{M_x}{M} = \frac{-3}{22}$

11. Lengths of human pregnancies are normally distributed with mean 268 days and standard deviation 15 days. What percentage of pregnancies last between 250 days and 280 days? Hint: use the normal probability density function: $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$

SOLN: $\frac{1}{15\sqrt{2\pi}}\int_{250}^{280} e^{-(x-268)^2/450}\,dx$

12. Let $f(x) = kx(1-x^2)$ if $0 \le x \le 1$ and $f(x) = 0$ if $x < 0$ or $x > 1$.

a. For what value of k is $f(x)$ a probability density function?

SOLN: $k\int_0^1 x(1-x^2)\,dx = k\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{k}{4} = 1 \Leftrightarrow k = 4$

b. For that value of k, find $P(X \ge 0)$

Clearly, this is 1.

c. Find the mean.

$$4\int_0^1 x^2(1-x^2)\,dx = 4\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{8}{15}$$